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THE DYNAMICS OF MAGNETIC FLOWS FOR ENERGIES ABOVE MAÑÉ'S CRITICAL VALUE

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ABSTRACT

We show that, for energies above Mañé's critical value, minimal magnetic geodesics are Riemannian $(A,0)$-quasi-geodesics where $A \to 1$ as the energy tends to infinity. As a consequence, on negatively curved manifolds, minimal magnetic geodesics lie in tubes around Riemannian geodesics.

Finally, we investigate a natural metric introduced by Mañé via the so-called action potential. Although this magnetic metric does depend on the magnetic field, the associated magnetic length turns out to be just the Riemannian length.

1. Introduction and results

A magnetic field is given by a closed 2-form Ω on a closed manifold M. We consider the magnetic flow on the universal cover $\widetilde{M}$; we will always assume that the pull-back of Ω on $\widetilde{M}$ is exact, i.e., the differential of a 1-form θ. Then the motion of a charged particle in the magnetic field is given by the Euler–Lagrange flow of the Lagrangian $L : T\widetilde{M} \to \mathbb{R}$ with

$$L(x,v) = \frac{1}{2}|v|_x^2 - \theta_x(v).$$

This can also be written as the Hamiltonian flow with

$$H(x,y) = \frac{1}{2}|y+\theta|_x^2.$$

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Hence, trajectories lie in level sets $\{H = k\}$ of fixed energy k. If the magnetic field vanishes this is precisely the geodesic flow on $\widetilde{M}$.

The aim of our present work is to understand the dynamics of magnetic flows, in particular its relation to the underlying geodesic flow. It is known that there is a critical energy level $c = c(L)$ at which a significant change of behaviour in the dynamics takes place. Indeed, this follows from Mañé's variational theory for convex Lagrangian systems (or Mather's theory of action-minimizing measures, respectively). The critical value $c(L)$ can be defined in two different ways, in either dynamical or geometric terms. In the first setting, $c(L)$ is the infimum of k's such that every closed loop in the universal cover has non-negative $(L+k)$-action. Geometrically, $c(L)$ is the infimum of energy values k such that the sublevel set $\{H < k\}$ contains an exact Lagrangian graph. It is a finite value if and only if θ can be chosen to be bounded. This implies that the flow on energy surfaces $E^{-1}(k)$ with $k > c(L)$ is a reparametrization of an appropriate Finsler geodesic flow.

The different characterizations of Mañé's critical value have been essential in describing the dynamics of magnetic flows for energies above the critical value; see, for example, [BP, CDI, CIPP]. We want to include, however, also a certain differential geometric flavour, by comparing the dynamics of the magnetic flow with that of the underlying geodesic flow, and by choosing formulations more in the language of classical differential geometry.

Our main results are the following:

Magnetic geodesics are Riemannian quasi-geodesics. For every energy value $k_0 > c(L)$, there exists a constant $A > 1$ such that minimal magnetic geodesics of energy $k \geq k_0$ are Riemannian $(A, 0)$-quasi-geodesics, with $A \to 1$ as $k_0 \to \infty$ (Theorem 2.9). The example of a constant magnetic field on the hyperbolic plane shows that the boundary value $c(L)$ is sharp.

The Morse-Lemma for magnetic geodesics. Assume that the manifold M has negative curvature. Then the classical Morse-Lemma states that (A, α)-quasi-geodesics lie in tubes around genuine geodesics. We prove here that the optimal width of these tubes goes to zero when (A, α) tends to $(1, 0)$ (Theorem 3.4).

Then the fact that magnetic geodesics are quasi-geodesics implies the Morse-Lemma for magnetic geodesics (Theorem 3.9), where the tubes can be chosen arbitrarily small if the magnetic field is weak enough (or, equivalently, the energy of the magnetic geodesic is sufficiently high).

The magnetic length is the Riemannian length. Using his so-called action potential, Mañé defined for each energy value k a pseudo-metric on the universal cover $\widetilde{M}$ which becomes a genuine metric when $k > c(L)$. This metric should encode properties of the magnetic field. Somewhat surprisingly, however, it turns out that the corresponding *length* does not contain any information about the magnetic field anymore—it is just the original Riemannian length (Theorem 4.3). This follows from a careful estimate for the difference between Mañé's and the Riemannian metric (Theorem 4.5).

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2. Magnetic geodesics are Riemannian quasi-geodesics

Let M be a closed manifold with universal covering $\pi\colon \widetilde{M} \to M$, and Ω a closed 2-form on M such that $\pi^*\Omega = d\theta$ for some 1-form θ on $\widetilde{M}$. If g is a Riemannian metric on M we denote by $|\cdot|_x$ the corresponding norm on the lifted tangent space $T_x\widetilde{M}$. Then the **magnetic Lagrangian** $L\colon T\widetilde{M} \to \mathbb{R}$ for the magnetic field given by θ is defined as

$$L(x,v) = \frac{1}{2}|v|_x^2 - \theta_x(v).$$

The corresponding Euler–Lagrange flow is called **magnetic flow** and its trajectories are **magnetic geodesics**, in contrast to **Riemannian geodesics**.[1] Note that L is convex and fibrewise superlinear, i.e., L restricted to each tangent space has positive definite Hessian and superlinear growth. Therefore we can define the **magnetic Hamiltonian** $H\colon T^*\widetilde{M} \to \mathbb{R}$ as the convex conjugate of L:

$$H(x,y) = \sup_{v \in T_x\widetilde{M}} [y(v) - L(x,v)].$$

PROPOSITION 2.1:

$$H(x,y) = \frac{1}{2}|y + \theta_x|_x^2.$$

Proof: Since L is convex and superlinear in each fibre, the **Legendre transformation** $\mathcal{L}_L\colon T\widetilde{M} \to T^*\widetilde{M}$ with

$$\mathcal{L}_L\colon (x,v) \mapsto (x, \partial_v L(x,v))$$

1 In the following, we will denote magnetic geodesics by the Greek letter γ, and Riemannian geodesics by the Latin c.

is a diffeomorphism. Moreover, we have $\partial_v L(x,v) = \langle v, \cdot\rangle_x - \theta_x(\cdot)$, so

$$y = \partial_v L(x,v) \iff v = (y+\theta_x)^\sharp.$$

Therefore,

$$\begin{aligned} H(x,y) &= (y(v) - L(x,v))|_{y=\partial_v L(x,v)} \\ &= y((y+\theta_x)^\sharp) - \frac{1}{2}|y+\theta_x|_x^2 + \theta_x((y+\theta_x)^\sharp) \\ &= \frac{1}{2}|y+\theta_x|_x^2. \qquad \blacksquare \end{aligned}$$

COROLLARY 2.2: *A magnetic geodesic γ has constant velocity.*

Proof: Since H is autonomous, it is constant along solutions. Therefore, the function

$$\begin{aligned} \partial_v L(\gamma(t),\dot\gamma(t))(\dot\gamma(t)) - L(\gamma(t),\dot\gamma(t)) &= \langle\dot\gamma(t),\dot\gamma(t)\rangle - \theta(\dot\gamma(t)) - \frac{1}{2}|\dot\gamma(t)|^2 + \theta(\dot\gamma(t)) \\ &= \frac{1}{2}|\dot\gamma(t)|^2 \end{aligned}$$

is constant. $\blacksquare$

The function $E\colon T\widetilde{M} \to \mathbb{R}$ with

$$E(x,v) = \frac{1}{2}|v|_x^2$$

is called the **energy**; it is constant along magnetic geodesics.

Let us now recall the notion of Mañé's critical value; we refer to [CDI, CI, CIPP, Mañ] for more details and references. Given any convex superlinear Lagrangian $L\colon T\widetilde{M} \to \mathbb{R}$, we define the **critical value** $c(L)$ as

$$c(L) = \inf\{k \in \mathbb{R} \mid \int_a^b (L(\gamma(t),\dot\gamma(t)) + k)dt \geq 0 \text{ for every closed curve } \gamma\colon [a,b] \to \widetilde{M}\}. \tag{1}$$

Remark 2.3: The critical value appears also in Mather's theory of minimizing measures [Mat]. Namely, for simply connected M, we have $c(L) = \alpha(0)$ where $\alpha\colon H^1(M,\mathbb{R}) \to \mathbb{R}$ is the convex conjugate of the minimal action.

We will make use of the following characterization of the critical value, proven by Burns and Paternain [BP, Appendix A].

THEOREM 2.4 ([CIPP, BP]):

$$c(L) = \inf_{f \in C^\infty(\widetilde{M},\mathbb{R})} \sup_{x\in\widetilde{M}} H(x,(df)_x).$$

This means that $c(L)$ is the infimum of energy values k such that the sublevel set $\{H < k\}$ contains an exact Lagrangian graph $\mathrm{gr}(df)$. The infimum is finite if and only if $\pi^*\Omega$ has a bounded primitive, i.e., if θ can be chosen to be bounded; see [BP].

Example 2.1: The critical value $c(L)$ is finite if the manifold M admits a Riemannian metric $\hat{g}$ (which need not be the original metric g) whose geodesic flow is an Anosov flow; see [BP]. This is the case, for instance, if $\hat{g}$ has negative curvature.

We have the following corollary; see [CIPP, CI] for a proof.

COROLLARY 2.6 ([CIPP]): *For $k > c(L)$, the magnetic flow on the energy level $E^{-1}(k)$ is a reparametrization of the geodesic flow of an appropriate Finsler metric F on $\widetilde{M}$.*

Recall that a Finsler metric on $\widetilde{M}$ is a continuous function $F: T\widetilde{M} \to [0,\infty)$, smooth away from the zero section, which is convex and positively 2-homogeneous on each fibre:

$$F(x,\lambda v) = \lambda^2 F(x,v) \quad \text{for all } \lambda > 0.$$

Moreover, it is shown in [CI] that the Finsler metric on the energy level $E^{-1}(k)$ can be chosen such that

$$\sqrt{F(x,v)} = L(x,v) + k - (df)_x(v) = 2k - (\theta + df)_x(v) \tag{2}$$

whenever $\mathrm{gr}(df) \subset \{H < k\}$.

Given two points $x, y \in \widetilde{M}$ and $k > c(L)$, we say that a curve $\gamma: [a,b] \to \widetilde{M}$ connecting them is **k-minimizing** if every other curve $\gamma': [a',b'] \to \widetilde{M}$ connecting x and y does not have smaller **$(L+k)$-action**:

$$\int_{a'}^{b'} (L(\gamma',\dot{\gamma}') + k)dt \geq \int_a^b (L(\gamma,\dot{\gamma}) + k)dt.$$

We point out that the time interval $[a',b']$ is free. It turns out, however, that the unique parametrization of a curve γ with least $(L+k)$-action satisfies $1/2|\dot{\gamma}|^2 = k$ [BP, Lemma 2.2]. It follows that k-minimizing curves are magnetic geodesics [CDI], so that we also call them **minimal magnetic geodesics**. A curve $\gamma: \mathbb{R} \to$

$\widetilde{M}$ is called a minimal magnetic geodesic if it is a minimal magnetic geodesic on each finite interval.

With this notation, it it apparent that, for fixed energy k, minimal magnetic geodesics are minimal Finsler geodesics for the Finsler metric F given above, because

$$\begin{aligned}\int_c^d \sqrt{F(\gamma,\dot\gamma)}dt &= \int_c^d (L(\gamma,\dot\gamma)+k-(df)_\gamma(\dot\gamma))dt\\ &= \int_c^d (L(\gamma,\dot\gamma)+k)dt \ + f(\gamma(c)) - f(\gamma(d))\end{aligned}$$

where $f(\gamma(c)) - f(\gamma(d)) = f(x) - f(y)$ is only a constant.

Remark 2.7: Magnetic fields can be defined without the exactness condition $\pi^*\Omega = d\theta$. Recall that Ω is a closed 2-form on M. Let ω_0 denote the symplectic form on TM, obtained by pulling back the standard symplectic form on T^*M via the Riemannian metric. We build the **twisted symplectic form** ω by setting

$$\omega = \omega_0 + p^*\Omega$$

with the projection $p\colon TM \to M$. Then the Hamiltonian flow of $E(x,v) = 1/2|v|_x^2$ with respect to the twisted symplectic form ω coincides with the magnetic flow defined above. However, we are interested in magnetic geodesics with minimal action, which can only be defined in the Lagrangian framework.

Remark 2.8: Up to now, we have fixed the magnetic field and varied the energy k. There is a dual viewpoint where one fixes the energy level but rescales the magnetic field instead. If we rescale the magnetic field via

$$\Omega_\lambda = \lambda\Omega$$

with $\lambda \geq 0$, and define **λ-magnetic geodesics** to be magnetic geodesics with respect to Ω_λ having energy $1/2|\dot\gamma(t)|^2 = 1/2$, then, after reparametrization, λ-magnetic geodesics with $\lambda = 1/\sqrt{2k}$ are magnetic geodesics with energy k. This observation allows one to switch between the following two situations:

1. magnetic geodesics with energy $k > c(L)$,
2. λ-magnetic geodesics with $\lambda < 1/\sqrt{2c(L)}$.

We define the **Lorentz force** $Y\colon TM \to TM$ by

$$\Omega_x(v,w) = g_x(Y_x(v),w).$$

This implies $\langle Y(v), v\rangle = 0$ for all v, and magnetic geodesics are solutions of the differential equation[2]

$$\nabla_{\dot\gamma}\dot\gamma = Y_\gamma(\dot\gamma).$$

The geodesic curvature of a magnetic geodesic γ of energy k is just the rescaled length of the Lorentz force:

$$|k_g(\gamma)| = \frac{1}{|\dot\gamma|^2}|Y_\gamma(\dot\gamma)| = \frac{1}{2k}|Y_\gamma(\dot\gamma)|. \tag{3}$$

Finally, we recall the notion of quasi-geodesics on a Riemannian manifold (X, g). A curve γ: $[a, b] \to X$ joining two points $x = \gamma(a)$ and $y = \gamma(b)$ in X is called an (A, α)-**quasi-geodesic** if

$$\frac{1}{A}|s-t| - \alpha \le d(\gamma(s), \gamma(t)) \le A|s-t| + \alpha$$

for all $s, t \in [a, b]$. Here, d is the distance with respect to the given Riemannian metric g. We point out that a quasi-geodesic need *not* be continuous. Finally, a curve γ: $\mathbb{R} \to X$ is an (A, α)-quasi-geodesic if it satisfies the above inequality for all $s, t \in \mathbb{R}$.

THEOREM 2.9: *Let M be any closed manifold, and $L(x, v) = 1/2|v|_x^2 - \theta_x(v)$ be a magnetic Lagrangian on $T\widetilde{M}$. Then, for each $k_0 > c(L)$, there is a constant $A = A(k_0) \in (1, \infty)$ such that every minimal magnetic geodesic with energy $k \ge k_0$ is, after reparametrization by arc-length, an $(A, 0)$-quasi-geodesic; moreover, $A(k_0) \to 1$ as $k_0 \to \infty$.*

Remark 2.10: Grognet [Gr, Prop. 3.1] showed that magnetic geodesics are quasi-geodesics, provided the manifold M has pinched negative curvature and the magnetic field is weak enough (in terms of the pinching constant). His proof uses geometric methods which work only in the negative curvature case. Our result holds in complete generality, and gives a sharp bound for the strength of the magnetic field (see Example 2.12 below).

Remark 2.11: Boyland and Golé [BG] consider the setting of time-dependent Lagrangians and prove that minimal trajectories are (λ, ϵ)-quasi-geodesics; see

2 By the way, this yields a more geometric proof of the fact that magnetic geodesics have constant energy:

$$\frac{d}{dt}\frac{1}{2}|\dot\gamma|^2 = \langle\nabla_{\dot\gamma}\dot\gamma, \dot\gamma\rangle = \langle Y_\gamma(\dot\gamma), \dot\gamma\rangle = 0.$$

[BG, Prop. 2.1]. However, in the time-dependent case, the proofs become much more technical, and–more important–the constants cannot be controlled. The number λ in [BG] does not tend to 1 as K (the equivalent of the energy) tends to infinity; moreover, ϵ is always non-zero.

Proof: Pick any minimal magnetic geodesic of energy $k \geq k_0 > c(L)$. According to Corollary 2.6 and (2), the magnetic flow on the energy surface $E^{-1}(k)$ is a reparametrization of a Finsler geodesic flow where

$$\sqrt{F(x,v)} = 2k - (\theta + df)_x(v).$$

For f we choose, once and for all, a function satisfying

$$H(x,(df)_x) < \frac{1}{2}(k_0 + c(L)),$$

which is possible due to Theorem 2.4. We rescale F by the factor $(2k)^{-1}$ and set

$$\hat{F} = \frac{1}{2k}F.$$

This does not affect any of the following arguments.

In order to prove the theorem, we claim that it suffices to show that there exists a constant $A > 1$ such that

$$\frac{1}{A} \cdot g_x(v,v) \leq \hat{F}(x,v) \leq A \cdot g_x(v,v) \tag{4}$$

for all (x,v) in the energy level $E^{-1}(k)$. More precisely, we claim that if (4) holds, then every minimal magnetic geodesic γ of energy k is an $(A,0)$-quasi-geodesic. Indeed, since every minimal magnetic geodesic is a minimal Finsler geodesic, we have

$$\begin{aligned} l(\gamma|_{[s,t]}) &= \int_s^t \sqrt{g_\gamma(\dot\gamma,\dot\gamma)}d\tau \leq \sqrt{A}\cdot \int_s^t \sqrt{\hat{F}(\gamma,\dot\gamma)}d\tau \\ &= \sqrt{A}\cdot \inf_{\bar\gamma} \int_s^t \sqrt{\hat{F}(\bar\gamma,\dot{\bar\gamma})}d\tau \leq A \cdot \inf_{\bar\gamma}\int_s^t \sqrt{g_{\bar\gamma}(\dot{\bar\gamma},\dot{\bar\gamma})}d\tau \\ &= A\cdot d(\gamma(s),\gamma(t)), \end{aligned}$$

so γ is an $(A,0)$-quasi-geodesic.

Note that, because of (2) and $|v|_x^2 = 2k$, (4) is equivalent to

$$\frac{1}{A} \leq \left(1 - \frac{(\theta + df)_x(v)}{2k}\right)^2 \leq A. \tag{5}$$

Define

$$A(k,k_0) = \left(1 - \sqrt{\frac{k_0 + c(L)}{2k}}\right)^{-2}.$$

We claim that (5) holds for $A = A(k, k_0)$. Indeed, our above choice of f yields

$$|\theta + df|_x < \sqrt{k_0 + c(L)}.$$

Moreover, we know that $|v|_x = \sqrt{2k}$. This implies

$$1 - \frac{(\theta + df)_x(v)}{2k} \geq 1 - \frac{|\theta + df|_x\sqrt{2k}}{2k} > 1 - \sqrt{\frac{k_0 + c(L)}{2k}}$$

as well as

$$1 - \frac{(\theta + df)_x(v)}{2k} \leq 1 + \frac{|\theta + df|_x\sqrt{2k}}{2k} < 1 + \sqrt{\frac{k_0 + c(L)}{2k}}.$$

Combining both inequalities we end up with

$$\begin{aligned}\frac{1}{A(k,k_0)} = \Big(1 - \sqrt{\frac{k_0 + c(L)}{2k}}\Big)^2 &\leq \Big(1 - \frac{(\theta + df)_x(v)}{2k}\Big)^2 \\ &\leq \Big(1 + \sqrt{\frac{k_0 + c(L)}{2k}}\Big)^2 \leq A(k, k_0).\end{aligned}$$

Finally, we define

$$A(k_0) = A(k_0, \min\{c(L) + 1, k_0\}).$$

Then is it clear that $A(k_0) \to 1$ as $k_0 \to \infty$. ∎

The constant $c(L)$ in the above theorem is the best possible in the sense that there is no smaller constant that works for *all* manifolds. This can be seen by the following example.

Example 2.12: Consider the constant magnetic field on the hyperbolic plane $\mathbb{H}^2$ given by the standard volume form

$$\Omega = \frac{dx \wedge dy}{y^2} = d\Big(\frac{dx}{y}\Big).$$

We point out that dx/y is a *bounded* primitive of Ω. The Lagrangian L is given by

$$L(x, v) = \frac{1}{2}|v|^2 - \frac{dx(v)}{y},$$

and the Lorentz force is

$$Y\Big(a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}\Big) = -b\frac{\partial}{\partial x} + a\frac{\partial}{\partial y}.$$

We claim that

$$c(L) = \frac{1}{2}. \tag{6}$$

This would follow from [BP, Thm. A] stating that $\sqrt{2c(L)}$ is equal to the inverse of Cheeger's isoperimetric constant; nevertheless, we include the following more elementary proof.

In order to show that $c(L) \leq 1/2$, we note that the $(L+k)$-action of a closed curve $\gamma(t) = (x(t), y(t))$ is given by

$$\begin{aligned}\int_\gamma (L+k)dt &= \frac{1}{2}\int_a^b \frac{\dot{x}^2 + \dot{y}^2 - 2\dot{x}y + 2ky^2}{y^2}dt \\ &= \frac{1}{2}\int_a^b \left[\left(\frac{\dot{x}-y}{y}\right)^2 + (2k-1) + \left(\frac{\dot{y}}{y}\right)^2\right]dt\end{aligned}$$

where the integrand is non-negative as long as $k \geq 1/2$. Hence the infimum $c(L)$ over all such k is at most $1/2$.

For the reversed inequality, we look at a hyperbolic circle γ of radius r. The length of γ is $2\pi \sinh r$, and the area of the enclosed disk B_r equals $2\pi(\cosh r - 1)$. Parametrizing γ such that the velocity is $\sqrt{2k}$, we obtain

$$\begin{aligned}\int_\gamma (L+k)dt &= 2k \cdot \frac{2\pi \sinh r}{\sqrt{2k}} - \int_\gamma \theta \\ &= 2\pi\sqrt{2k}\sinh r - \int_{B_r} \Omega \\ &= 2\pi(\sqrt{2k}\sinh r - (\cosh r - 1)) \\ &\leq 2\pi(1 - (1-\sqrt{2k})\cosh r).\end{aligned}$$

If $k < 1/2$, the last expression goes to $-\infty$ if $r \to \infty$. Hence $c(L) \geq 1/2$, and the proof of (6) is finished.

Geometrically, it is quite clear that magnetic geodesics of energy $k \leq 1/2$ cannot be $(A,0)$-quasi-geodesics. Namely, we look at λ-magnetic geodesics with $\lambda \geq 1/\sqrt{2c(L)} = 1$; we know from (3) that λ is the geodesic curvature. Thus, magnetic geodesics for $\lambda \geq 1$ are horocycles resp. circles, which can never be $(A,0)$-quasi-geodesics (because any geodesic connects two different points at infinity).

Summarizing, this example shows that the constant $c(L)$ in Theorem 2.9 is indeed sharp.

3. The Morse-Lemma for magnetic geodesics

In this section, we suppose that the closed manifold M has negative curvature. Then $\widetilde{M}$ is diffeomorphic to $\mathbb{R}^n$, so our standing assumption that $\pi^*\Omega$ be exact is always satisfied. Our aim is to prove that, as long as the magnetic field is weak enough (given in terms of the critical value $c(L)$), minimal magnetic geodesics in $\widetilde{M}$ lie near Riemannian geodesics. To do so, we consider the more general case of (A,α)-quasi-geodesics, for minimal magnetic geodesics of energy above Mañé's critical value $c(L)$ are $(A,0)$-quasi-geodesics (Theorem 2.9).

Let us recall some notions from geometry. The Hausdorff distance of two closed sets $B, C \subset \widetilde{M}$ is defined as

$$d_H(B,C) = \inf\{a \in (0,\infty) \mid B \subset U_a(C) \text{ and } C \subset U_a(B)\}$$

where $U_a(B)$ denotes the open a-neighbourhood around B. The geometric **boundary at infinity** $\widetilde{M}(\infty)$ is defined as the set of equivalence classes of geodesic rays, where two rays are said to be equivalent if they have bounded Hausdorff distance. Topologically, $\widetilde{M}(\infty)$ is a sphere, and we define

$$\overline{M} = \widetilde{M} \cup \widetilde{M}(\infty).$$

For more details concerning the boundary at infinity we refer to [Eb].

In this section, we will always assume that geodesics are parametrized by arc-length.

It is well known for complete, simply connected manifolds $\widetilde{M}$ with negative upper curvature bound that any two points at infinity $p, q \in \widetilde{M}(\infty)$ can be joined by a unique Riemannian geodesic c_{pq}. We will show an analogous existence result for minimal magnetic geodesics joining $p, q \in \widetilde{M}(\infty)$ for every fixed energy $k_0 > c(L)$.

PROPOSITION 3.1: *Suppose $\widetilde{M}$ is the universal covering of a closed manifold M with negative sectional curvature, L a magnetic Lagrangian, and $k_0 > c(L)$. Then any two points $p, q \in \widetilde{M}(\infty)$ can be joined by a minimal magnetic geodesic γ_{pq} of energy k_0.*

Proof: Take the Riemannian geodesic c_{pq} connecting p and q, and choose minimal magnetic geodesics γ_n connecting pairs of points $p_n, q_n \in c_{pq}(\mathbb{R}) \subset \widetilde{M}$ converging to p and q, respectively; this is possible due to a theorem by Mañé for coverings [CI, Prop. 3.5.1]. According to Theorem 2.9, the γ_n are quasi-geodesics, so the classical Morse-Lemma (see, for instance, [CDP]) implies that there is a $D > 0$ such that $d_H(\gamma_n, c_{p_n q_n}) < D$ for all n. Now choose a compact ball $B \subset \widetilde{M}$

of radius D around some "central" point on c_{pq} between p_1 and q_1. Then each of the γ_n must intersect B, i.e., there is a tangent vector v_n of γ_n which lies in TB; note that $|v_n| = \sqrt{2k_0}$ for all n. By compactness of B, we can assume that $v_n \to v$ as $n \to \infty$. Then the magnetic geodesic γ with initial condition v will stay in a $2D$-tube around c_{pq}, so it connects its end points p and q.

It remains to prove that γ is minimal. If the restriction γ: $[a,b] \to \widetilde{M}$ to some finite interval was not minimal we could find another magnetic geodesic γ': $[a',b'] \to \widetilde{M}$ with the same end points such that

$$\int_{a'}^{b'} (L(\gamma',\dot\gamma') + k_0)dt < \int_a^b (L(\gamma,\dot\gamma) + k_0)dt.$$

Since γ is obtained as a limit of the γ_n, we can find $a_n, b_n \in \mathbb{R}$ such that $\gamma_n(a_n) \to \gamma(a)$ and $\gamma_n(b_n) \to \gamma(b)$. Since

$$\lim_{n\to\infty} \int_{a_n}^{b_n} (L(\gamma_n,\dot\gamma_n) + k_0)dt = \int_a^b (L(\gamma,\dot\gamma) + k_0)dt,$$

one would obtain, for large enough n, a strictly smaller $(L+k)$-action by replacing the curve $\gamma_n|_{[a_n,b_n]}$ by the concatenation of a short arc connecting $\gamma_n(a_n)$ with $\gamma(a) = \gamma'(a')$, the curve γ', and a short arc connecting $\gamma'(b') = \gamma(b)$ with $\gamma_n(b_n)$. But this would contradict the minimality of γ_n. ∎

The following theorem describes the so-called stability of quasi-geodesics.

THEOREM 3.2 (Classical Morse-Lemma): *Let $\widetilde{M}$ be a complete, simply connected Riemannian manifold with curvature $K_{\widetilde{M}} \le -k^2 < 0$, and $A \ge 1, \alpha \ge 0$ be real numbers. Then there is a constant $D = D(k, A, \alpha) > 0$ such that the following holds. For any two points $p, q \in \widetilde{M}(\infty)$, the Hausdorff distance between the image of the geodesic connecting p and q and the image of any (A,α)-quasi-geodesic with the same end points is at most D.*

Remark 3.3: This theorem has an analogue in the more general context of hyperbolic metric spaces in the sense of Gromov; see, for instance, [CDP]. For a proof of the classical Morse-Lemma we refer to [Kn].

In the following, we will show that the optimal tube width D for bi-infinite (A,α)-quasi-geodesics tends to 0 as $(A,\alpha) \to (1,0)$. This fact might be intuitively clear. However, the upper bounds for the tube width given in [BH, CDP, Eb, Kn] do *not* tend to 0 as $(A,\alpha) \to (1,0)$.

The precise result is described in Theorem 3.4 below. For a fixed Riemannian manifold $\widetilde{M}$ and given $A > 1, \alpha > 0$, we define

$$\begin{aligned} D_0(A,\alpha) = \inf\{D > 0 \mid \forall (A,\alpha)\text{-quasi-geodesics } \gamma\colon \mathbb{R} \to \widetilde{M} \\ \exists \text{ geodesic } c\colon \mathbb{R} \to \widetilde{M} \text{ with } d_H(\gamma(\mathbb{R}), c(\mathbb{R})) \leq D\}, \end{aligned} \tag{7}$$

i.e., $D_0(A,\alpha) \in [0,\infty]$ is the **optimal tube width** for (A,α)-quasi-geodesics. Note that, in the negative curvature case, the classical Morse-Lemma implies that $D_0(A,\alpha)$ is finite for every pair (A,α).

THEOREM 3.4: *Let $\widetilde{M}$ be a complete, simply connected Riemannian manifold with $K_{\widetilde{M}} \leq -k^2 < 0$. Let $D_0(A,\alpha)$ be the optimal tube width defined in (7). Then*

$$\lim_{(A,\alpha)\to(1,0)} D_0(A,\alpha) = 0.$$

Remark 3.5: In the particular case when $\alpha = 0$, Bangert and Lang [BL] proved a quantitative version of Theorem 3.4 with the optimal constant $D_0(A,0) = \pi\sqrt{A^2-1}/2k$. Their proof is based on estimates for quasi-minimizing submanifolds, in contrast to our approach which deals with curves alone.

Before proving the theorem, let us first draw our main conclusion from it. For minimal magnetic geodesics, we define the optimal tube width $D_1(k_0)$ for energies at least k_0 as

$$\begin{aligned} D_1(k_0) = \inf\{D > 0 \mid \forall \text{min. magn. geodesics } \gamma\colon \mathbb{R} \to \widetilde{M} \text{ of energy } \geq k_0 \\ \exists \text{ geodesic } c\colon \mathbb{R} \to \widetilde{M} \text{ with } d_H(\gamma(\mathbb{R}), c(\mathbb{R})) \leq D\}. \end{aligned} \tag{8}$$

By Theorem 2.9, minimal magnetic geodesics of energy $k_0 > c(L)$ are $(A(k_0), 0)$-quasi-geodesics, which implies the following result.

THEOREM 3.6: *Let L be a magnetic Lagrangian on the universal covering $\widetilde{M}$ of a closed Riemannian manifold M with negative curvature. Then, for each energy level $k_0 > c(L)$, the optimal tube width $D_1(k_0)$ is finite and satisfies*

$$\lim_{k_0\to\infty} D_1(k_0) = 0.$$

Remark 3.7: We note that Grognet [Gr] also proved a version of the Morse-Lemma for magnetic flows (without the result for the optimal tube width, however). It differs from our version insofar as his proof and his energy bound are given in geometric terms, whereas ours are more related to dynamics.

Proof: The finiteness statement is an immediate consequence of the classical Morse-Lemma (Theorem 3.2) and Theorem 2.9. Since, again by Theorem 2.9, minimal magnetic geodesics of energy at least k_0 are $(A(k_0), 0)$-quasi-geodesics, we conclude for the tube widths defined in (7) and (8) that

$$D_1(k_0) = D_0(A(k_0), 0).$$

Now the assertion that $\lim_{k_0\to\infty} D_1(k_0) = 0$ follows from Theorem 3.4 and the fact that $\lim_{k_0\to\infty} A(k_0) = 1$. ∎

Proof of Theorem 3.4: The theorem is proven by contradiction. Since $D_0(A, \alpha)$ is a non-decreasing function of A and α, we assume that

$$\lim_{(A,\alpha)\to(1,0)} D_0(A, \alpha) = \delta > 0.$$

Our aim is to prove that there are $A_1 > 1, \alpha_1 > 0$ with $D_0(A_1, \alpha_1) < \delta$, which is a contradiction.

We fix, once and for all,

$$A_0 = 4/3,$$

$\alpha_0 > 0$, and a $D > 0$ such that all (A_0, α_0)-quasi-geodesics $\gamma: \mathbb{R} \to \widetilde{M}$ lie within D-tubes around geodesics $c_0: \mathbb{R} \to \widetilde{M}$. This is possible according to Theorem 3.2. The proof of Theorem 3.4 is based on the following three lemmata.

LEMMA A: *Let $\gamma: \mathbb{R} \to \widetilde{M}$ be an (A_0, α_0)-quasi-geodesic, and $l > 12\alpha_0, t \in \mathbb{R}$ be given. Let $c: \mathbb{R} \to \widetilde{M}$ be the geodesic through $\gamma(t - l)$ and $\gamma(t + l)$, i.e., $c(a) = \gamma(t - l)$ and $c(b) = \gamma(t + l)$. Then we have*

$$P_c(\gamma(t)) \in c([a, b]),$$

where $P_c: \widetilde{M} \to c(\mathbb{R})$ denotes the orthogonal projection.

The statement of Lemma A is illustrated in Figure 1.

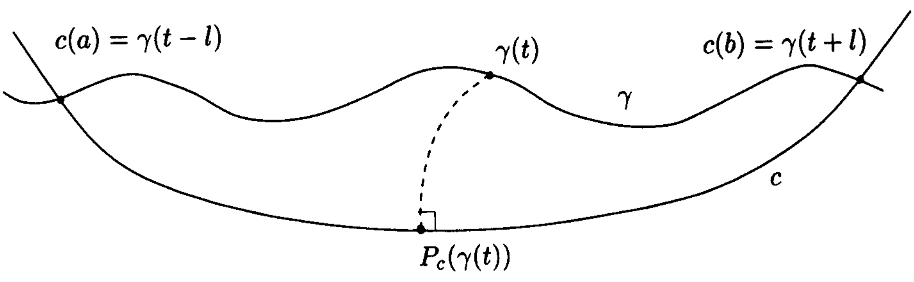

Figure 1. Illustration of Lemma A

The next two lemmata contain the main steps in the proof of Theorem 3.4. Roughly speaking, they state the following. Given an (A, α)-quasi-geodesic γ there is, according to the classical Morse-Lemma, some geodesic c_0 in a D-neighbourhood of $\gamma(\mathbb{R})$. Now, for any point $\gamma(t)$, choose two points $\gamma(t \pm l_1)$ very far outside, i.e., with l_1 very large. Consider the geodesic c through these two points, and project the mid-point $\gamma(t)$ onto c. Then, on the one hand, Lemma B states that the distance between $P_c(\gamma(t))$ and $c_0(\mathbb{R})$ becomes arbitrarily small if we take l_1 large enough. On the other hand, Lemma C asserts that the distance between $\gamma(t)$ and $P_c(\gamma(t))$ can be made arbitrarily small by choosing (A, α) very close to $(1,0)$. Taken together, this means that the geodesic c_0 lies not only in a D-tube around γ, but its distance to γ can be made arbitrarily small by letting (A, α) tend to $(1,0)$. Therefore $D_0(A, \alpha)$ can be made as small as we wish, which is exactly what we want to prove.

LEMMA B: *For every $d > 0$ there is an $l_1 = l_1(d) > 0$ with the following property. Let $\gamma: \mathbb{R} \to \widetilde{M}$ be an (A_0, α_0)-quasi-geodesic, t a real number, and $c: \mathbb{R} \to \widetilde{M}$ the geodesic through $\gamma(t - l_1)$ and $\gamma(t + l_1)$. Let $c_0: \mathbb{R} \to \widetilde{M}$ be the geodesic with $\gamma(\mathbb{R}) \subset U_D(c_0(\mathbb{R}))$.*[3] *Then*

$$d(P_c(\gamma(t)), c_0(\mathbb{R})) \leq d.$$

LEMMA C: *For every $d > 0$ and $l_1 > 0$ there is a pair $(A_1, \alpha_1) \in (1, A_0] \times (0, \alpha_0]$ with the following property. Let $\gamma: \mathbb{R} \to \widetilde{M}$ be an (A_1, α_1)-quasi-geodesic, t a real number, and $c: \mathbb{R} \to \widetilde{M}$ be the geodesic through $\gamma(t - l_1)$ and $\gamma(t + l_1)$. Then*

$$d(\gamma(t), P_c(\gamma(t)) \leq d.$$

For the sake of clarity, we postpone proving the three lemmata and continue with the proof of Theorem 3.4.

Recall that we want to prove the existence of an (A_1, α_1) with $D_0(A_1, \alpha_1) < \delta$. Let $l_1 = l_1(\delta/4)$ and (A_1, α_1) be given as in Lemmata B and C with $d = \delta/4$; without loss of generality, we can also assume that $\alpha_1 < \delta/4$. Then, for every (A_1, α_1)-quasi-geodesic $\gamma: \mathbb{R} \to \widetilde{M}$ and every $t \in \mathbb{R}$, we have

$$d(\gamma(t), P_{c_0}(\gamma(t))) \leq d(\gamma(t), P_c(\gamma(t))) + d(P_c(\gamma(t)), c_0(\mathbb{R})) \leq \frac{\delta}{4} + \frac{\delta}{4} = \frac{\delta}{2}, \tag{9}$$

where c and c_0 are the geodesics described in Lemma B. This implies $\gamma(\mathbb{R}) \subset U_{3\delta/4}(c_0(\mathbb{R}))$.

3 Note that c_0 is uniquely defined by this property. The constant D has been chosen above.

On the other hand, we want to prove that $c_0(\mathbb{R}) \subset U_{3\delta/4}(\gamma(\mathbb{R}))$. Notice that the projection P_{c_0} does not increase distances. Therefore, for every $c(t)$, there is a $c(t')$ with $|t-t'| \le \alpha_1$ which is the projection of some point $\gamma(\tau)$. In view of (9), we have $d(c(t),\gamma(\tau)) \le d(c(t),c(t')) + d(c(t'),\gamma(\tau)) \le \alpha_1 + \delta/2 < 3\delta/4$, and this finishes the proof of the theorem. ∎

It remains to prove the three lemmata above.

Proof of Lemma A: Since $\gamma\colon \mathbb{R} \to \widetilde{M}$ is an (A_0,α_0)-quasi-geodesic with $A_0 = 4/3$, we conclude that

$$d(c(a),c(b)) = d(\gamma(t-l),\gamma(t+l)) \ge 2l/A_0 - \alpha_0 = 3l/2 - \alpha_0.$$

Moreover, since the orthogonal projection is distance non-increasing, we also have

$$d(c(a),P_c(\gamma(t))) \le d(\gamma(t-l),\gamma(t)) \le A_0 l + \alpha_0 = 4l/3 + \alpha_0.$$

Since $l > 12\alpha_0$, we have $3l/2 - \alpha_0 > 4l/3 + \alpha_0$ and hence $d(c(a),c(b)) \ge d(c(a),P_c(\gamma(t)))$.

Similarly, one obtains $d(c(a),c(b)) \ge d(c(b),P_c(\gamma(t)))$. This implies $P_c(\gamma(t)) \in c([a,b])$. ∎

Proof of Lemma B: For the proof we use a sublemma; see Figure 2 for illustration.

SUBLEMMA: *For every $d > 0$ there is a $q = q(d) > 0$ with the following property. Let $c_0\colon \mathbb{R} \to \widetilde{M}$ be a geodesic, and $a,b \in \mathbb{R}$ with $b - a > 2q$. Let $c\colon \mathbb{R} \to \widetilde{M}$ be the geodesic through two points y_1, y_2 with $y_1 \in U_D(c_0(a))$ and $y_2 \in U_D(c_0(b))$. Then we have for all $z_0 \in c(\mathbb{R})$ with $P_{c_0}(z_0) \in c_0([a+q,b-q])$ that*

$$d(z_0,c_0(\mathbb{R})) = d(z_0,P_{c_0}(z_0)) \le d.$$

Proof: Since M is compact the curvature $K_{\widetilde{M}}$ satisfies

$$K_{\widetilde{M}} \le -\kappa^2 < 0$$

for a suitable $\kappa > 0$. Let $d > 0$ be given. We show that the sublemma holds for any choice $q > D$ satisfying

$$\frac{q+3D}{q-D} < \cosh(\kappa d). \tag{10}$$

Let c_0, c and a, b with $b - a > 2q$ be as in the sublemma. Define the numbers a_0, b_0 by $y_1 = c(a_0)$ and $y_2 = c(b_0)$, and set

$$V = \{t \in [a_0, b_0] \mid P_{c_0}(c(t)) \in c_0([a+q, b-q])\},$$
$$t_1 = \min V = \min\{t \in V \mid P_{c_0}(c(t)) = c_0(a+q)\},$$
$$t_2 = \max V = \max\{t \in V \mid P_{c_0}(c(t)) = c_0(b-q)\}.$$

Note that V might be disconnected, so we only have $V \subset [t_1, t_2]$. One observes that $d(P_{c_0}(y_1), c_0(a)) < D$ and $d(P_{c_0}(y_2), c_0(b)) < D$, since the projection P_{c_0} is distance non-increasing.

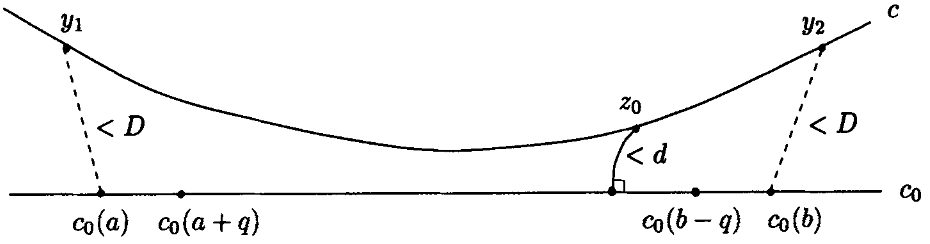

Figure 2. Illustration of the sublemma

Below, we are going to show the existence of numbers $s_1 \in [a_0, t_1]$ and $s_2 \in [t_2, b_0]$ such that

$$d(P_{c_0}(c(s_i)), c(s_i)) < d \quad \text{for } i = 1, 2. \tag{11}$$

The convexity of the distance function implies

$$d(P_{c_0}(t), c(t)) < d \quad \text{for all } t \in [s_1, s_2].$$

Since $V \subset [s_1, s_2]$, this will then finish the proof of the sublemma.

The existence of a number $s_1 \in [a_0, t_1]$ satisfying (11) is proven by contradiction. Assume that

$$\min_{t \in [a_0, t_1]} d(c(t), P_{c_0}(c(t))) \geq d.$$

Then a well-known Jacobi field estimate (see the appendix) yields

$$d(y_1, c(t_1)) \geq \cosh(\kappa d) d(P_{c_0}(y_1), c_0(a+q)) \geq \cosh(\kappa d)(q - D).$$

Since $P_{c_0}(c(t_1)) = c_0(a+q)$ this implies

$$\begin{aligned} 2D + b - a &\geq d(y_1, y_2) \\ &\geq d(y_1, c(t_1)) + d(c(t_1), y_2) \\ &\geq (q-D)\cosh(\kappa d) + d(P_{c_0}(c(t_1)), P_{c_0}(y_2)) \\ &= (q-D)\cosh(\kappa d) + d(c_0(a+q), P_{c_0}(y_2)) \\ &\geq (q-D)\cosh(\kappa d) + (b-D) - (a+q). \end{aligned}$$

Consequently, we end up with

$$\cosh(\kappa d) \leq \frac{q+3D}{q-D},$$

contradicting our choice of $q > D$. The existence of an $s_2 \in [t_2, b_0]$ satisfying (11) is proven similarly. ∎

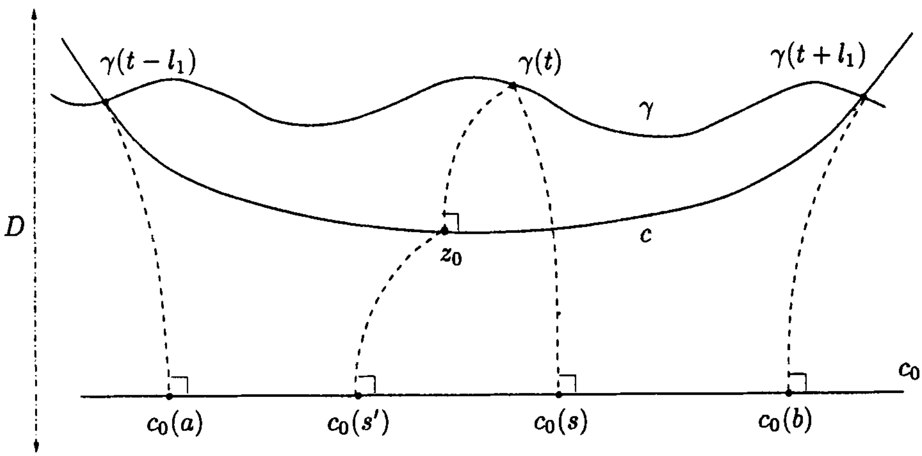

Figure 3. Illustration of the proof of Lemma B

We continue with the proof of Lemma B. Given $d > 0$, we find a $q = q(d)$ as asserted in the sublemma. Having this, we pick any l_1 which is large enough such that

$$l_1/A_0 - \alpha_0 - 6D \geq q.$$

Now define two numbers $a < b$ by $P_{c_0}(\gamma(t-l_1)) = c_0(a), P_{c_0}(\gamma(t+l_1)) = c_0(b)$, and s, s' by $P_{c_0}(\gamma(t)) = c_0(s), P_{c_0}(z_0) = c_0(s')$ where $z_0 = P_c(\gamma(t))$; see Figure 3 for an illustration. Since γ is an (A_0, α_0)-quasi-geodesic we have

$$s - a = d(c_0(a), c_0(s)) \geq d(\gamma(t-l_1), \gamma(t)) - 2D \geq l_1/A_0 - \alpha_0 - 2D.$$

Note that, by convexity of the distance function, the geodesic segment of c connecting $\gamma(t-l_1)$ with $\gamma(t+l_1)$ lies completely in the D-tube around $c_0(\mathbb{R})$. According to Lemma A, z_0 lies on this segment. Thus we conclude

$$\begin{aligned}|s-s'| &= d(c_0(s), c_0(s')) \\ &\le d(c_0(s), \gamma(t)) + d(\gamma(t), z_0) + d(z_0, c_0(s')) \\ &\le D + 2D + D = 4D.\end{aligned}$$

Putting everything together we obtain

$$s' - a \ge s - a - |s-s'| \ge l_1/A_0 - \alpha_0 - 2D - 4D = l_1/A_0 - \alpha_0 - 6D \ge q$$

and, similarly, $b - s' \ge q$. This implies that $P_{c_0}(z_0) = c_0(s') \in c_0([a+q, b-q])$, and Lemma B follows from the sublemma. ∎

Proof of Lemma C: Without loss of generality, we assume that $d < l_1$. Let γ, c, t, and l_1 be as in the lemma, where we will specify (A_1, α_1) later. For simplicity, we introduce the following additional notions:

$$\begin{array}{llll} y_1 = \gamma(t-l_1), & y_2 = \gamma(t+l_1), & z = \gamma(t), & z_0 = P_c(z), \\ a_1 = d(y_1, z_0), & a_2 = d(z_0, y_2), & d_0 = d(z, z_0). & \end{array}$$

We want to prove that

$$d_0 \le d.$$

Without loss of generality, we can assume that $a_1 \ge a_2$. On the one hand, we obtain by the first law of cosine (see, e.g., [BGS, p. 7]) that

$$(A_1 l_1 + \alpha_1)^2 \ge d(y_1, z)^2 \ge a_1^2 + d_0^2.$$

On the other hand, since γ is an (A_1, α_1)-quasi-geodesic, we find

$$2a_1 \ge a_1 + a_2 = d(y_1, y_2) \ge 2l_1/A_1 - \alpha_1.$$

Combining both inequalities yields

$$d_0^2 \le (A_1 l_1 + \alpha_1)^2 - a_1^2 \le (A_1 l_1 + \alpha_1)^2 - (l_1/A_1 - \alpha_1/2)^2.$$

Choosing (A_1, α_1) sufficiently near to $(1, 0)$ we can make the right hand side smaller than d^2. This finishes the proof of Lemma C. ∎

Remark 3.8: Note that a more careful discussion of the previous considerations would even allow one to derive an explicit upper bound for $D_0(A, \alpha)$ in Theorem 3.4.

The following theorem summarizes the results of this section for minimal magnetic geodesics.

THEOREM 3.9 (Morse-Lemma for magnetic geodesics): *Let L be a magnetic Lagrangian on the universal covering $\widetilde{M}$ of a closed Riemannian manifold with negative curvature, and $k_0 > c(L)$ be a fixed energy level. Then there is a constant*[4] *$D_1(k_0) > 0$ such that the following holds.*

Any two points $p, q \in \widetilde{M}(\infty)$ can be joined by a minimal magnetic geodesic with energy k_0, and all minimal magnetic geodesics connecting p and q lie within a $D_1(k_0)$-tube around the unique Riemannian geodesic between p and q. Moreover, we have

$$\lim_{k_0 \to \infty} D_1(k_0) = 0.$$

4. The magnetic length is the Riemannian length

Given a metric space (X, d), the length of a continuous curve c: $[a, b] \to X$ is defined as

$$l(c) = \sup_{a=t_0<t_1<\cdots<t_N=b} \sum_{j=0}^{N-1} d(c(t_j), c(t_{j+1})).$$

The **inner metric** d_{int} associated to d is then defined by

$$d_{\text{int}}(x, y) = \inf\{l(c) \mid c\colon [a, b] \to X, c(a) = x, c(b) = y\}.$$

Obviously, $d_{\text{int}} \geq d$ and $(d_{\text{int}})_{\text{int}} = d_{\text{int}}$. A metric space (X, d) with $d = d_{\text{int}}$ is called a **length space**.

Now suppose, as before, that the magnetic Lagrangian $L\colon T\widetilde{M} \to \mathbb{R}$ is given by $L(x, v) = 1/2|v|_x^2 - \theta_x(v)$. Then the so-called **action potential** Φ_k is defined as

$$\Phi_k(x, y) = \inf\{\int_a^b (L(\gamma(t), \dot\gamma(t)) + k)dt \mid \gamma\colon [a, b] \to \widetilde{M}, \gamma(a) = x, \gamma(b) = y\}.$$

We emphasize that the time interval $[a, b]$ is free. In the case when $k > c(L)$, Φ_k satisfies the triangle inequality $\Phi_k(x, y) + \Phi_k(y, z) \leq \Phi_k(x, z)$ so that

(12) $$d^{\text{magn}}(x, y) = d_k^{\text{magn}}(x, y) = \frac{1}{2\sqrt{2k}}(\Phi_k(x, y) + \Phi_k(y, x))$$

4 $D_1(k_0)$ has been defined in (8).

is a metric.[5] We refer to this metric as (the rescaled) **Mañé's metric**; see [Mañ, CDI]. d^{magn} is invariant under the group of deck transformations. The length associated to Mañe's metric d^{magn} is denoted by l^{magn} and the usual length associated to the Riemannian metric d^{Riem} is denoted by l^{Riem}; note that $d^{\text{Riem}}_{\text{int}} = d^{\text{Riem}}$.

Let us calculate Mañé's metric for a particular example in order to show that $d^{\text{magn}} \neq d^{\text{Riem}}$.

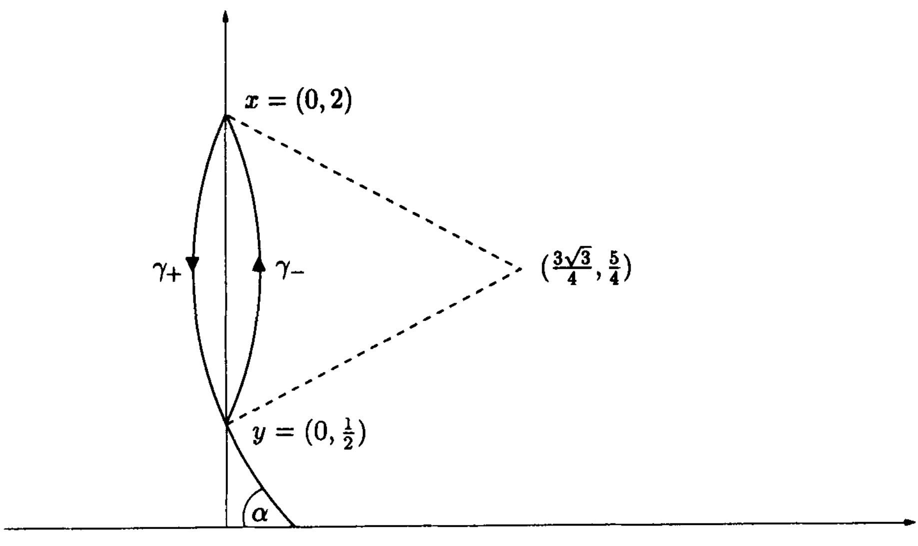

Figure 4. Mañé's metric in the hyperbolic plane

Example 4.1: We consider the constant magnetic field given by the 1-form $\theta = dx/y$ on the hyperbolic plane $\mathbb{H}^2$; see Example 2.12. Let $x = (0, 2)$ and $y = (0, 1/2)$. Then

$$d^{\text{Riem}}(x, y) = \ln 4 \approx 1.38629436.$$

The minimal magnetic geodesics $\gamma_\pm$ connecting x and y, respectively, y and x are Euclidean circle segments. With an appropriate choice of the energy k (compare (13)), we can assume that their radius is 3/2; see Figure 4. Then the angle α satisfies $\cos\alpha = \sqrt{11}/6$, so the geodesic curvature of $\gamma_\pm$ is given by

$$\lambda = \cos\alpha = \sqrt{11}/6.$$

5 Note that $d^{\text{magn}}(x, y) \geq 0$, with $d^{\text{magn}}(x, y) > 0$ for $x \neq y$, is an immediate consequence of (1).

Moreover, the energy of $\gamma_\pm$ is

$$k = \frac{1}{2\lambda^2} = \frac{18}{11} > \frac{1}{2} = c(L). \tag{13}$$

In view of the symmetry of $\gamma_\pm$, the Mañé metric can be calculated as

$$\begin{aligned} d^{\mathrm{magn}}(x,y) &= l^{\mathrm{Riem}}(\gamma_+) - \frac{\sqrt{11}}{6}\int_{\gamma_+}\theta \\ &= \int_{5\pi/6}^{7\pi/6}\Big[\frac{\sqrt{\dot x(t)^2+\dot y(t)^2}}{y(t)} - \frac{\sqrt{11}}{6}\frac{\dot x(t)}{y(t)}\Big]dt \\ &= \frac{\sqrt{11}}{18}\pi + \Big(\frac{3}{2} - \frac{5\sqrt{11}}{24}\Big)\int_{5\pi/6}^{7\pi/6}\frac{dt}{5/4+3/2\sin t}. \end{aligned}$$

Here, we have parametrized the curve γ_+ by

$$(x(t),y(t)) = \Big(\frac{3\sqrt{3}}{4},\frac{5}{4}\Big) + \frac{3}{2}(\cos t,\sin t), \quad \frac{5\pi}{6} \le t \le \frac{7\pi}{6}.$$

Evaluating the (explicitly solvable) integral shows indeed that

$$d^{\mathrm{magn}}(x,y) \approx 1.36634949 \neq 1.38629436 \approx d^{\mathrm{Riem}}(x,y).$$

Although the two metrics d^{magn} and d^{Riem} are different, they induce the same topologies. In fact, the two metrics are equivalent, as the following proposition shows.

PROPOSITION 4.2: *For $k > c(L)$, we have*

$$\frac{k-c(L)}{4k}d^{\mathrm{Riem}} \le d^{\mathrm{magn}} \le d^{\mathrm{Riem}}.$$

Proof: Given two points $x,y \in \widetilde{M}$, we consider the minimal Riemannian geodesic $c_+\colon [a,b] \to \widetilde{M}$ from x to y with $1/2|\dot c_+(t)|^2 = k$, as well as its counterpart c_- from y to x (which is just c_+ traversed backwards). It follows from

$$\begin{aligned} d^{\mathrm{Riem}}(x,y) &= \frac{1}{2}\Big[\int_a^b |\dot c_+(t)|dt + \int_a^b |\dot c_-(t)|dt\Big] \\ &= \frac{1}{2\sqrt{2k}}\Big[\int_a^b 2k dt + \int_a^b 2k dt\Big] \\ &= \frac{1}{2\sqrt{2k}}\Big[\int_a^b (L(c_+(t),\dot c_+(t)) + k)dt + \int_a^b (L(c_-(t),\dot c_-(t)) + k)dt\Big] \end{aligned}$$

and (12) that

$$d^{\mathrm{Riem}}(x,y) \ge d^{\mathrm{magn}}(x,y). \tag{14}$$

Let γ_+ and γ_- be the minimal magnetic geodesics connecting x and y, respectively, y and x. Then $\gamma = \gamma_+ \cup \gamma_-: [a,b] \to \widetilde{M}$ is a closed curve. Set

$$k_0 = \frac{k + c(L)}{2}.$$

Then

$$\begin{aligned} d^{\mathrm{magn}}(x,y) &= \frac{1}{2\sqrt{2k}} \int_a^b (L(\gamma,\dot\gamma) + k)dt \\ &= \frac{1}{2\sqrt{2k}} \int_a^b (L(\gamma,\dot\gamma) + k_0)dt + \frac{k-k_0}{2\sqrt{2k}}(b-a) \\ &\geq 0 + \frac{k-c(L)}{4\sqrt{2k}}(b-a) \quad \text{since } k_0 > c(L) \\ &= \frac{k-c(L)}{8k} l^{\mathrm{Riem}}(\gamma) \quad \text{since } |\dot\gamma| = \sqrt{2k} \\ &\geq \frac{k-c(L)}{4k} d^{\mathrm{Riem}}(x,y). \qquad \blacksquare \end{aligned}$$

The following result states the somewhat surprising fact that, although Mañé's metric is different from the Riemannian metric (see Example 4.1), the length associated to Mañé's metric does *not* contain any information about the original magnetic field-it is just the Riemannian length.

THEOREM 4.3: *Consider a magnetic Lagrangian $L(x,v) = 1/2|v|_x^2 - \theta_x(v)$ on the universal covering $\widetilde{M}$ of a closed manifold M. Then, for energy values $k > c(L)$, we have*

$$l^{\mathrm{magn}} = l^{\mathrm{Riem}}$$

on $\widetilde{M}$. In particular, the magnetic inner metric $d_{\mathrm{int}}^{\mathrm{magn}}$ coincides with the Riemannian metric.

Remark 4.4: Let us emphasize that we do *not* assume that M has negative curvature.

Theorem 4.3 is a consequence of the following stronger result for Mañé's and the Riemannian metric themselves.

THEOREM 4.5: *Consider a magnetic Lagrangian $L(x,v) = 1/2|v|_x^2 - \theta_x(v)$ on the universal covering $\widetilde{M}$ of a closed manifold M. Then, for energy values $k > c(L)$, we have*

$$d^{\mathrm{magn}} - d^{\mathrm{Riem}} = O(d^2)$$

as $d \to 0$, where d can stand for either d^{Riem} or d^{magn}.

Remark 4.6: The essential feature for the proof (see below) is that d^{magn} is the distance d^F of a Finsler metric F on $\widetilde{M}$ of the form

$$F(x,v) = \left(\sqrt{g_x(v,v)} + \beta_x(v)\right)^2,$$

where β is a 1-form on $\widetilde{M}$ with bounded differential. Therefore, Theorem 4.5 holds also in this slightly more general framework.

Before proving Theorem 4.5, let us show how Theorem 4.3 follows from it.

Proof of Theorem 4.3: This is a proof for general metric spaces. Let us assume that we have two metrics d, d' on some set X, satisfying

$$d - d' = O(d^2)$$

as $d \to 0$, as well as

$$d - d' = O({d'}^2)$$

as $d' \to 0$. Note that these assumptions imply that both metrics induce the same topology.

Let $c\colon [a,b] \to X$ be a continuous curve. Then, given any $\epsilon > 0$, we obtain for every sufficiently fine partition of c:

$$\begin{aligned}\sum_{j=0}^{N-1} d(c(t_j), c(t_{j+1})) &\leq \sum_{j=0}^{N-1} [d'(c(t_j), c(t_{j+1})) + C \cdot d'(c(t_j), c(t_{j+1}))^2] \\ &= \sum_{j=0}^{N-1} (1 + Cd'(c(t_j), c(t_{j+1}))) \cdot d'(c(t_j), c(t_{j+1})) \\ &\leq (1 + C\epsilon) \sum_{j=0}^{N-1} d'(c(t_j), c(t_{j+1})),\end{aligned}$$

where C is some positive constant independent of ϵ. This shows that $l \leq l'$, and the reversed inequality follows from interchanging d and d'. ∎

For the proof of Theorem 4.5, we need an isoperimetric inequality for spanning disks. This is, of course, well known; compare [BZ, Thm. 17.2.3] where the existence of a smooth spanning disk is proven. However, our aim is to construct an immersed disk, except for one point. For the convenience of the reader, we present the short proof here.[6]

We start with a preliminary lemma. Let us denote $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$.

6 We do not care about *optimal* isoperimetric constants.

LEMMA 4.7: *Let $\gamma\colon \mathbb{S}^1 \to \mathbb{R}^n, n \geq 3$, be an embedded circle. Suppose $p \in \mathbb{R}^n$ and $l > 0$ are such that $\gamma(\mathbb{S}^1) \subset B_l(p)$. Then there exists a point $q \in B_{4l}(p)$ such that the union of the line segments connecting q with $\gamma(t)$ is an immersed closed disk F satisfying $\partial F = \gamma(\mathbb{S}^1)$ and*

$$\operatorname{area}(F) \leq \frac{25}{2} \operatorname{length}(\gamma) \cdot l.$$

Proof: Set $A = \bigcup_{t\in\mathbb{S}^1}(\gamma(t) + \mathbb{R}\dot{\gamma}(t)) \subset \mathbb{R}^n$. Pick any point $q_0 \in \mathbb{R}^n$ with $|q_0 - p| = 3l$. Since A has codimension at least one, there is a point $q \in B_l(q_0)\backslash A$. Then the map

$$\phi\colon [0,1] \times \mathbb{S}^1 \to \mathbb{R}^n, \quad \phi(s,t) = (1-s)q + s\gamma(t) \tag{15}$$

defines an immersion of $(0,1)\times\mathbb{S}^1$ into $\mathbb{R}^n$; moreover, we have $\gamma(\mathbb{S}^1) = \phi(\{1\}\times\mathbb{S}^1)$. Define

$$F = \phi((0,1) \times \mathbb{S}^1).$$

We want to estimate area(F) from above. To do so, we project the curve γ, seen from the reference point q, onto the unit sphere by setting

$$\Psi\colon \mathbb{S}^1 \to S^{n-1}(0), \quad \Psi(t) = \frac{1}{|\gamma(t) - q|}(\gamma(t) - q).$$

Then

$$\operatorname{length}(\gamma) \geq l \cdot \operatorname{length}(\Psi) \tag{16}$$

since $\gamma(\mathbb{S}^1)$ lies outside the ball $B_l(q)$. On the other hand, we have

$$F \subset \tilde{F} := q + \bigcup_{t\in\mathbb{S}^1} (0, 5l) \cdot \Psi(t)$$

since $\gamma(\mathbb{S}^1)$ is contained in the ball $B_{5l}(q)$. An elementary calculation yields

$$\operatorname{area}(F) \leq \operatorname{area}(\tilde{F}) = \operatorname{length}(\Psi) \cdot \frac{(5l)^2}{2} = \frac{25}{2} \operatorname{length}(\Psi) \cdot l^2,$$

so (16) gives the desired result. ∎

PROPOSITION 4.8: *Given an embedding $\gamma\colon \mathbb{S}^1 \to \mathbb{R}^n, n \geq 2$, there is an immersed closed disk F such that $\partial F = \gamma(\mathbb{S}^1)$ and*

$$\operatorname{area}(F) \leq \frac{25}{2} \operatorname{length}(\gamma)^2.$$

Proof: For $n \geq 3$, this is just Lemma 4.7 if we choose $p \in \gamma(\mathbb{S}^1)$ and $l = \operatorname{length}(\gamma)$. The assertion for $n = 2$ follows from the classical isoperimetric inequality. ∎

COROLLARY 4.9: *Let $\gamma: \mathbb{S}^1 \to \mathbb{R}^n, n \geq 2$, be an embedding and $\Omega = d\theta$ a bounded 2-form on $\mathbb{R}^n$. Then*

$$\Big|\int_\gamma \theta\Big| \leq \frac{25}{2}\|\Omega\|_\infty \cdot \mathrm{length}(\gamma)^2.$$

Proof: Let ϕ be the map defined in (15), and apply Stokes' Theorem to the immersed annulus $\phi([\epsilon,1]\times\mathbb{S}^1)$ with arbitrarily small $\epsilon > 0$. ∎

Now we are ready to present the proof of Theorem 4.5.

Proof of Theorem 4.5: Let us denote by γ_+ and γ_- the minimal magnetic geodesics connecting x and y, respectively, y and x. Note that none of the curves $\gamma_\pm$ can have self-intersections for, otherwise, it would contain a closed loop which would yield a positive contribution in the $(L+k)$-action,[7] and one could obtain a curve with strictly smaller $(L+k)$-action by removing this loop. In addition, perhaps after an arbitrarily small C^1-perturbation, we may assume that γ_+ and γ_- intersect transversally (in particular, they intersect in finitely many points). For the sake of simplicity, we assume further that γ_+ and γ_- intersect only at their common end points x and y; this is no loss of generality, because, otherwise, the following estimates are valid for each of the finitely many simply closed loops formed by $\gamma_\pm$.

With these simplifications,the magnetic distance of x and y is given by

$$d^{\mathrm{magn}}(x,y) = \frac{1}{2}[l(\gamma_+) + l(\gamma_-)] - \frac{1}{2\sqrt{2k}}\int_\gamma \theta$$

where $l = l^{\mathrm{Riem}}$ is the Riemannian length and $\gamma = \gamma_+ \cup \gamma_-$ is a simply closed curve; the first equality follows from $|\dot\gamma_\pm|^2 = 2k$. In view of Proposition 4.2, we obtain

$$\begin{aligned} 0 &\geq d^{\mathrm{magn}}(x,y) - d^{\mathrm{Riem}}(x,y) \\ &= \frac{1}{2}[l(\gamma_+) + l(\gamma_-) - 2d^{\mathrm{Riem}}(x,y)] - \frac{1}{2\sqrt{2k}}\int_\gamma \theta \qquad (17) \\ &\geq -\frac{1}{2\sqrt{2k}}\int_\gamma \theta. \end{aligned}$$

We would now like to apply Corollary 4.9 which was, however, formulated for embedded circles in $\mathbb{R}^n$, and not for circles with corners in a manifold. The corners at the two end points x, y can be smoothened by an arbitrarily small C^1-perturbation. Moreover, by choosing $d^{\mathrm{Riem}}(x,y)$ small enough we may work

7 Here we use the assumption that $k > c(L)$, with $c(L)$ given by (1).

in a sufficiently small coordinate chart around some point z_0 and choose normal coordinates such that

$$g_{ij}(z) = \delta_{ij} + O(|z - z_0|^2).$$

Then the length and the area for the metric g in that chart can be compared to those for the flat metric:

$$(1-\epsilon)\,\mathrm{length}_g \leq \mathrm{length}_{\mathrm{flat}} \leq (1+\epsilon)\,\mathrm{length}_g$$
$$(1-\epsilon)\,\mathrm{area}_g \leq \mathrm{area}_{\mathrm{flat}} \leq (1+\epsilon)\,\mathrm{area}_g\,.$$

Therefore, the error we make in applying Corollary 4.9 can be made arbitrarily small. Ignoring this error, we find a constant $C = C(k, g, \|\Omega\|_\infty) > 0$ such that

$$\frac{1}{2\sqrt{2k}}\left|\int_\gamma \theta\right| \leq C \cdot l(\gamma)^2.$$

Since minimal magnetic geodesics are Riemannian $(A, 0)$-quasi-geodesics (Theorem 2.9), we end up with the estimate

$$\frac{1}{2\sqrt{2k}}\left|\int_\gamma \theta\right| \leq C' \cdot (d^{\mathrm{Riem}}(x,y))^2, \tag{18}$$

provided $d^{\mathrm{Riem}}(x, y)$ is small enough.

Finally, combining (17) and (18), we find that

$$-C' \cdot (d^{\mathrm{Riem}}(x,y))^2 \leq d^{\mathrm{magn}}(x,y) - d^{\mathrm{Riem}}(x,y) \leq 0$$

for sufficiently small $d^{\mathrm{Riem}}(x, y)$, which implies

$$d^{\mathrm{magn}} - d^{\mathrm{Riem}} = O((d^{\mathrm{Riem}})^2).$$

By Proposition 4.2, we also have

$$d^{\mathrm{magn}} - d^{\mathrm{Riem}} = O((d^{\mathrm{magn}})^2).$$

This finishes the proof of the theorem. ∎

Appendix

In this appendix we include, for the convenience of the reader, the proof of the following well known fact. Let X be a complete, simply connected manifold of strictly negative curvature:

$$K_X \leq -\kappa^2 < 0.$$

Consider a geodesic c_0 in X, as well as two points $p, q \in X$, such that the connecting geodesic segment c_{pq}: $[a,b] \to X$ stays outside some neighbourhood of $c_0(\mathbb{R})$:

$$d(c_{pq}([a,b]), c_0(\mathbb{R})) \geq d > 0.$$

Then we must have

$$d(p,q) \geq \cosh(\kappa d) \cdot d(P_{c_0}(p), P_{c_0}(q)),$$

where P_{c_0}: $X \to c_0(\mathbb{R})$ is the orthogonal projection onto c_0.

Obviously, this result follows immediately by integration if we can show that

$$|DP_{c_0}(x)(v)| \leq \frac{1}{\cosh(\kappa d(x, P_{c_0}(x))}|v| \tag{19}$$

for all $x \in X$ and $v \in T_xX$.

For the proof of (19) we introduce an appropriate geodesic variation α. Let c: $(-\epsilon,\epsilon) \to X$ be a curve satisfying $\dot{c}(0) = v$ and σ: $(-\epsilon,\epsilon) \to \mathbb{R}$ be defined by $P_{c_0}(c(s)) = c_0(\sigma(s))$. Then there is a uniquely defined variation α: $(-\epsilon,\epsilon) \times [0,1] \to X$ such that

$$\alpha(s,0) = c_0(\sigma(s)), \quad \alpha(s,1) = c(s),$$

as well as

$$t \mapsto \alpha(s,t) \text{ is a geodesic for all } s \in (-\epsilon,\epsilon).$$

The corresponding variational field $J(t) = \frac{\partial\alpha}{\partial s}(0,t)$ is a Jacobi field along the geodesic $\alpha_0(t) = \alpha(0,t)$ with

$$\langle J(0), \dot{\alpha}_0(0)\rangle = 0 = \langle J(0), J'(0)\rangle.$$

Note that $d(x, P_{c_0}(x)) = |\dot{\alpha}_0|$ and that (19) is equivalent to

$$|J(1)| \geq |J(0)| \cosh(\kappa|\dot{\alpha}_0|). \tag{20}$$

Now decompose the Jacobi field J into normal and tangential components with respect to the curve α_0. $J(0)$ is purely normal, and if the normal component of $J(t)$ (which is a Jacobi field itself) satisfies (20) then the full Jacobi field will do so, too. Therefore, we may assume that J is a normal Jacobi field.

Let $f(t) = |J(t)|$. Then $f(0) = |J(0)|$, and $f'(0) = 0$ because of $\langle J'(0), J(0)\rangle = 0$. An easy calculation yields the following differential inequality for f:

$$\begin{aligned} f'' &= \frac{1}{|J|}(\langle J, J''\rangle + |J'|^2) - \frac{1}{|J|^3}\langle J, J'\rangle^2 \\ &\geq \kappa^2|\dot{\alpha}_0|^2 f. \end{aligned}$$

Let

$$f_0(t) = |J(0)| \cosh(\sqrt{\kappa}|\dot{\alpha}_0|t) \tag{21}$$

be the solution of the corresponding differential equation $f_0'' = \kappa^2|\dot{\alpha}_0|^2 f_0$. We conclude from $f_0 > 0$ that the function $g = f'f_0 - ff_0'$ satisfies

$$g' = f''f_0 - ff_0'' \geq \kappa|\dot{\alpha}_0|^2(ff_0 - ff_0) = 0.$$

Together with $g(0) = 0$, this implies $g \geq 0$. If we define $h = f/f_0$ we obtain

$$h' = \frac{g}{f_0^2} \geq 0.$$

Because of $h(0) = 1$ and $h' \geq 0$ we conclude $h \geq 0$, i.e., $f \geq f_0$. Putting this together with (21) yields the required estimate (20), which finishes the proof.

References

[BGS] W. Ballmann, M. Gromov and V. Schroeder, *Manifolds of Nonpositive Curvature*, Birkhäuser, Basel, 1985.

[BL] V. Bangert and U. Lang, *Trapping quasiminimizing submanifolds in spaces of negative curvature*, Commentarii Mathematici Helvetici **71** (1996), 122–143.

[BG] P. Boyland and C. Golé, *Lagrangian systems on hyperbolic manifolds*, Ergodic Theory and Dynamical Systems **19** (1999), 1157–1173.

[BH] M. R. Bridson and A. Haefliger, *Metric Spaces of Non-Positive Curvature*, Springer, Berlin, 1999.

[BZ] Y. D. Burago and V. A. Zalgaller, *Geometric Inequalities*, Springer, Berlin, 1988.

[BP] K. Burns and G. Paternain, *Anosov magnetic flows, critical values and topological entropy*, Nonlinearity **15** (2002), 281–314.

[CDI] G. Contreras, J. Delgado and R. Iturriaga, *Lagrangian flows: the dynamics of globally minimizing orbits II*, Boletim da Sociedade Brasileira de Matemática, Nova Série **28** (1997), 155–196.

[CIPP] G. Contreras, R. Iturriaga, G. Paternain and M. Paternain, *Lagrangian graphs, minimizing measures and Mañé's critical values*, Geometric and Functional Analysis **8** (1998), 788–809.

[CI] G. Contreras and R. Iturriaga, *Global Minimizers of Autonomous Lagrangians*, 22nd Brazilian Mathematics Colloquium, IMPA, 1999.

[CDP] M. Coornaert, T. Delzant and A. Papadopoulos, *Géométrie et théorie des groupes*, Lecture Notes in Mathematics **1441**, Springer, Berlin, 1990.

[Eb] P. Eberlein, *Geodesic flow in certain manifolds without conjugate points*, Transactions of the American Mathematical Society **167** (1972), 151–170.

[Gr] S. Grognet, *Flots magnétiques en courbure négative*, Ergodic Theory and Dynamical Systems **19** (1999), 413–436.

[Kn] G. Knieper, *Hyperbolic dynamics and Riemannian geometry*, in *Handbook of Dynamical Systems*, Vol. 1A (B. Hasselblatt and A. Katok, eds.), Elsevier, Amsterdam, 2002, pp. 453–546.

[Mañ] R. Mañé, *Lagrangian flows: the dynamics of globally minimizing orbits*, Boletim da Sociedade Brasileira de Matemática, Nova Série **28** (1997), 141–153.

[Mat] J. N. Mather, *Action minimizing invariant measures for positive definite Lagrangian systems*, Mathematische Zeitschrift **207** (1991), 169–207.